Traveling Wave Solutions for the Foam Drainage Equation and the Enhanced (G'/G)-Expansion Method

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Abstract: In this paper, the enhanced (G'/G)-expansion method is used to assemble the traveling wave solutions involving parameters of the Foam Drainage equation, where $G = G(\xi)$ satisfies a second order linear differential equation. The traveling wave solutions are expressed by the hyperbolic functions and the trigonometric functions. When some arbitrary functions included in these solutions are taken as some special functions, these solutions possess profuse structures. This method is direct, concise elementary and effective and can be used for many other nonlinear evolutions equations.

Keywords: enhanced (G'/G)-expansion method, Foam Drainage equation, NLEEs, Traveling wave solutions, Solitary waves.

1. Introduction:

The innovation of the soliton, its remarkable properties and the incredible richness of structure are all included in its mathematical description. The story begins with the observation by Jhon Scott Russell of "the great wave of translation" He states:

"Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears [1]".

The appearance of solitary wave solutions in nature is quite common. References can be made to Bell-Shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in fluids, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions etc. The traveling wave solutions of the Korteweg-de Varies (KdV) and Boussinesq equations, which describe water waves, are famous examples as well.

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for examples [2-26]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented such as the extended tanh-function method [2, 3], the tanh-function method [4, 5, 6, 7], the F-function expansion method [8, 9], the exp-function expansion method [10, 11, 12, 13], the Jacobi elliptic function expansion method [14, 15], the Backlund transform [16],The homogeneous balance method [17], the homotopy analysis method [18, 19], the (G'/G)-expansion method [20, 21, 22, 23], the exp(-Φ(ξ))-expansion method [24, 25, 26] and so on.

In this paper, we contemplate the foam drainage equation and apply to the enhanced (G'/G)-expansion method. Foams are of huge significant in many technological processes and applications and their properties are subjects in rigorous studies from both sensible and scientific point of view. Foams are general in foods and personal care products such as lotions and creams and foams over and over again arise for the period of clearing of clothes and scrubbing [27]. There are now a lot of applications of polymeric foam [28] and more newly metallic foams which are foams throughout metals such as aluminum [29]. We give attention to a quantitative explanation of the coupling of drainage. Foam drainage is the flow of liquid through plateau borders and intersections of four channels between the bubbles driven by gravity and capillarity.

The paper is organized as follows. In section 2, we describe briefly the enhanced (G'/G)-expansion method, where $G = G(\xi)$ satisfies a second order linear differential equation $G'' + \lambda G = 0$, where $\xi = k(x + \omega t)$, where k, ω and λ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. In section 3, we apply this method to the foam drainage equation. In section 4, we discussed the physical elucidation and graphical demonstration. Finally, in section 5, some conclusions are given.

2. Description of the enhanced (G'/G)-expansion method:

In this section, we portray in niceties the enhanced (G'/G)-expansion method for outcome traveling wave equations of nonlinear equations. Any nonlinear equation in two independent variables x and t can be articulated in the following form:

$$\Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}.........) = 0, \qquad (2.1)$$

where $u(\xi) = u(x,t)$ is an anonymous function, $\Re$ is a polynomial of $u(x,t)$ and its partial derivatives wherein the uppermost order derivatives and non-linear stipulations are concerned. The following steps are concerned in result the solution of nonlinear Equation (2.1) using this method.

Step 1: The given PDE (2.1) can be distorted into ODE using the transformation $\xi = x \pm \omega t$, where ω is the speed of traveling wave such that $\omega \in R - \{0\}$.

The traveling wave alteration permits us to reduce Eq. (2.1) to the following ODE:

$$\Psi(u, u', u'', \ldots\ldots\ldots\ldots) = 0, \qquad (2.2)$$

Where Ψ is a polynomial in $u(\xi)$ and its derivatives, where

$$u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2 u}{d\xi^2}, \text{ and so on.}$$

Step 2: Now, we expect that the Eq. (2.2) has a general solution of the form

$$(2.3)$$

$$u(\xi) = \sum_{i=n}^{n} \left(\frac{a_i (G'/G)^i}{(1 + \lambda (G'/G))^i} + b_i (G'/G)^{i-1} \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu}\right)} \right),$$

subject matter to the condition that

$G = G(\xi)$ satisfy the equation

$$G'' + \lambda G = 0, \qquad (2.4)$$

Where $a_i, b_i (-n \le i \le n; n \in N)$, and λ are constant to be determined, provided that $\sigma = \pm 1$ and $\mu \neq 0$.

Step 3: The positive integer n can be determined by balancing the highest order derivatives to the highest order nonlinear terms appear in Eq. (2.1) or in Eq. (2.2). More accurately, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives respond to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, \quad D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q), \quad (2.5)$$

Step 4: We substitute Eq. (2.3) into Eq. (2.2) and use Eq. (2.4). We then collect all the coefficient of $(G'/G)^j$ and

$(G'/G)^j \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu}\right)}$ together. Since Eq. (2.3) is a

solution of Eq. (2.2). we can set each of the coefficient equal to zero which leads to a system of algebraic equations in terms of

$a_i, b_i (-n \le i \le n; n \in N)$, λ and ω . One can solves easily these system equations using Maple.

Step 5: For $\mu < 0$ general solution of Eq. (2.4) gives

$$\frac{G'}{G} = \sqrt{-\mu} \tanh\left(A + \sqrt{-\mu}\ \xi\right), \qquad (2.6)$$

and $\qquad \frac{G'}{G} = \sqrt{-\mu} \coth\left(A + \sqrt{-\mu}\ \xi\right), \qquad (2.7)$

And for $\mu > 0$, we get

$$\frac{G'}{G} = \sqrt{\mu} \tan\left(A - \sqrt{\mu}\ \xi\right), \qquad (2.8)$$

and $\dfrac{G'}{G} = \sqrt{\mu} \cot\left(A + \sqrt{\mu}\ \xi\right), \qquad (2.9)$

where A is an arbitrary constant. Finally we can construct a number of families of travelling wave solutions of Eq. (2.1) by substituting the values of $a_i, b_i (-n \le i \le n; n \in N)$, λ and ω (obtained in Step 3) and using Eq. (2.6) to Eq. (2.9) into Eq. (2.3).

3. Application:

In this section, we pertain the enhanced (G'/G)-expansion method to put up the traveling wave solutions for foam drainage equation

$$\psi_t + \left(\psi^2 - \frac{\sqrt{\psi}}{2} \psi_x\right)_x = 0, \qquad (3.1)$$

Where ψ is the cross segment of a channel formed where three films meet, usually indicate as plateau border and x and t are scaled position and time coordinates respectively. By means of the traveling wave variable $\psi = \varphi(\xi)$, $\xi = k(x + \omega t)$, where k and ω are constant, Eq. (3.1) is altered into the following ordinary differential equation (ODE) for $u = u(\xi)$.

$$k \omega \varphi' + k \left(\varphi^2 - \frac{k}{2} \sqrt{\varphi}\ \varphi'\right)' = 0, \qquad (3.2)$$

Where prime denote the differentiation with regard to ξ . Now, integrating Eq. (3.2) with respect to ξ and bearing in mind the zero constant for integrating we have,

$$k \omega \varphi + k \left(\varphi^2 - \frac{k}{2} \sqrt{\varphi}\ \varphi'\right) = 0, \qquad (3.3)$$

Afterwards we use the transformation

$$\varphi(\xi) = u^2(\xi),\tag{3.4}$$

to exchange Eq.(3.3) to

$$k\,\omega u^2 + k\,u^4 - k^2\,u^2\,u' = 0,\tag{3.5}$$

or equivalently

$$\omega + u^2 - k\,u' = 0,\tag{3.6}$$

Now, balancing the highest order derivative of u' and nonlinear term of u^2 in Eq.(3.6), we obtain $n = 1$.

So the equation (3.6) takes the following solution

$$u(\xi) = a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_1(1+\lambda(G'/G))}{(G'/G)} + b_0(G'/G)^{-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$$

$$+ b_1\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)},\tag{3.7}$$

where $G = G(\xi)$ satisfy the Eq. (2.4). Substituting Eq. (3.7) hooked on the Eq. (3.6) and using Eq. (2.4), we get a polynomial in $(G'/G)^j$ and $(G'/G)^j \sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$.

Surroundings the coefficient of $(G'/G)^j$ and $(G'/G)^j \sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$ equal to zero, we obtain a system containing a bulky number of algebraic equations in terms of unknown coefficients. We have solved this system of equations using Maple 13 and obtained the following set of solutions:

Set   1.:
$$\omega = k^2\mu,\ a_0 = -k\,\mu\,\lambda,\ a_1 = 0,\ a_{-1} = k\,\mu,\ b_0 = 0,\ b_1 = 0,\ b_{-1} = 0,$$

Set   2.:
$$\omega = k^2\mu,\ a_0 = k\,\mu\,\lambda,\ a_1 = -k\,\mu\,\lambda^2 - k,\ a_{-1} = 0,\ b_0 = 0,\ b_1 = 0,\ b_{-1} = 0,$$

Set   2.:
$$\omega = \frac{1}{4}k^2\mu,\ a_0 = -\frac{1}{2}k\,\mu\,\lambda,\ a_1 = 0,\ a_{-1} = \frac{1}{2}k\,\mu,\ b_0 = \pm\frac{1}{2}\frac{\mu k}{\sqrt{\sigma}},\ b_1 = 0,\ b_{-1} = 0,$$

Substituting Set 1-Set 3 into Eq. (3.7) along with Eq. (2.6)-Eq. (2.9) and using the transformation $\psi(x,t) = \varphi(\xi) = u^2(\xi)$, we have the following families of traveling wave solutions:

Hyperbolic function solutions:

when $\mu < 0$, we get the following three families of hyperbolic function solutions.

Family 1: $\psi_1(x,t) = \left(\dfrac{k\,\mu}{\sqrt{-\mu}}\coth(A+\sqrt{-\mu}\,\xi)\right)^2$,

$$\psi_2(x,t) = \left(\frac{k\,\mu}{\sqrt{-\mu}}\tanh(A+\sqrt{-\mu}\,\xi)\right)^2,$$

where $\xi = k(x+k^2\mu\,t)$,

Family 2: $\psi_3(x,t) = \left(\dfrac{k\,\mu\,\lambda - k\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\,\xi)}{1+\lambda\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\,\xi)}\right)^2$,

$$\psi_4(x,t) = \left(\frac{k\,\mu\,\lambda - k\sqrt{-\mu}\coth(A+\sqrt{-\mu}\,\xi)}{1+\lambda\sqrt{-\mu}\coth(A+\sqrt{-\mu}\,\xi)}\right)^2,$$

where $\xi = k(x+k^2\mu\,t)$,

Family 3: $\psi_{5,6}(x,t) = \left(\dfrac{k\sqrt{\mu}}{2I}(\coth(A+\sqrt{-\mu}\,\xi)\pm\csc h(A+\sqrt{-\mu}\,\xi))\right)^2$,

$$\psi_{7,8}(x,t) = \left(\frac{k\sqrt{\mu}}{2I}(\tanh(A+\sqrt{-\mu}\,\xi)\pm I\sec h(A+\sqrt{-\mu}\,\xi))\right)^2,$$

where $\xi = k(x+\frac{1}{4}k^2\mu\,t)$,

Trigonometric function solutions:

when $\mu > 0$, we get the following three families of trigonometric function solutions.

Family 4: $\psi_9(x,t) = \left(k\sqrt{\mu}\cot(A-\sqrt{\mu}\,\xi)\right)^2$,

$$\psi_{10}(x,t) = \left(k\sqrt{\mu}\tan(A-\sqrt{\mu}\,\xi)\right)^2,$$

where $\xi = k(x+k^2\mu\,t)$,

Family 5: $\psi_{11}(x,t) = \left(\dfrac{k\,\mu\,\lambda - k\sqrt{\mu}\tan(A-\sqrt{\mu}\,\xi)}{1+\lambda\sqrt{\mu}\tan(A-\sqrt{\mu}\,\xi)}\right)^2$,

$$\psi_{12}(x,t) = \left(\frac{k\,\mu\,\lambda - k\sqrt{\mu}\cot(A+\sqrt{\mu}\,\xi)}{1+\lambda\sqrt{\mu}\cot(A+\sqrt{\mu}\,\xi)}\right)^2,$$

where $\xi = k(x+k^2\mu\,t)$,

Family 6: $\psi_{13,14}(x,t) = \left(\dfrac{k\sqrt{\mu}}{2}(\cot(A-\sqrt{\mu}\,\xi)\pm\csc(A+\sqrt{\mu}\,\xi))\right)^2$,

$$\psi_{15,16}(x,t) = \left(\frac{k\sqrt{\mu}}{2}(\tan(A+\sqrt{\mu}\,\xi)\pm\sec(A+\sqrt{\mu}\,\xi))\right)^2,$$

where $\xi = k(x + \dfrac{1}{4}k^2\mu t)$,

Remark: All the obtained solutions have been tartan with maple by putting them back into the original equations and found correct.

4. Results and Discussion:

In this section, we will portray the physical elucidation and graphical demonstration of the solutions of the Foam Drainage equation.

4.1. Physical Elucidation:

A quantum of energy or quasi particle that can be propagate as a traveling wave in nonlinear systems and is neither preceded nor followed by another such disturbance; does not obey the superposition principle and does not dissipate. Soliton waves can travel long distances with little loss of energy or structure. Solitary waves are localized travelling waves travelling with constant speeds and shape, asymptotically zero at large distances. Solitons are special kinds of solitary waves. The soliton solution is especially localized solution, hence $u'(\xi), u''(\xi)$ and $u'''(\xi) = 0$ as with $\xi = \pm\infty, \xi = x - ct$. Solitons have a remarkable soliton property in that it keeps its identity upon interchanging with other solutions. Solitons solutions are give rise to particle like structures, such as magnetic monopoles etc. So, soliton are everywhere in the nature. The solution $\psi_1(x,t)$ is represented the soliton solution. The Fig. 1 has been shown the shape of the solution $\psi_1(x,t)$ for $\mu = -1, k = 0.5, \ A = 0$ with $-10 \le x, t \le 10$. For the fixed values $\mu = -1, \ k = 1, \ A = 0$ with $-3 \le x, t \le 3$, solution $\psi_2(x,t)$ represents the exact solitary wave solutions of bell type which is shown in Fig. 2. For the fixed values $\mu = 1, \ k = 1, \ A = 0$ with $-3 \le x, t \le 3$, solution $\psi_{10}(x,t)$ represents periodic wave solutions. Periodic solutions are traveling wave solutions are periodic such as $\cos(x-t)$. The exact periodic traveling wave solution is represented graphically in Fig. 3. Finally, solution $\psi_5(x,t)$ is singular soliton of the Foam Drianage equation. The Fig. 4 shows the shape of the singular soliton solution, obtain from $\psi_5(x,t)$ for $\mu = -1, \ k = 1, \ A = 0$ with $-3 \le x, t \le 3$.

4.2. Graphical Demonstration:

In this sub-section, the graphical demonstration of the solutions is given below in the figures (Fig. 1-4) with the aid of mathematical software Maple 13.
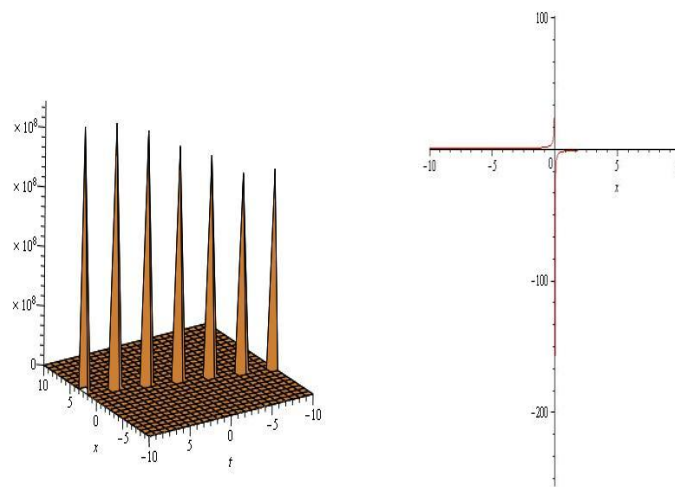

Fig. 1. Soliton profile of Foam Drianage equation for $\mu = -1, \ k = 0.5, \ A = 0$ with $-10 \le x, t \le 10$. (Only shows the shape of $\psi_1(x,t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.
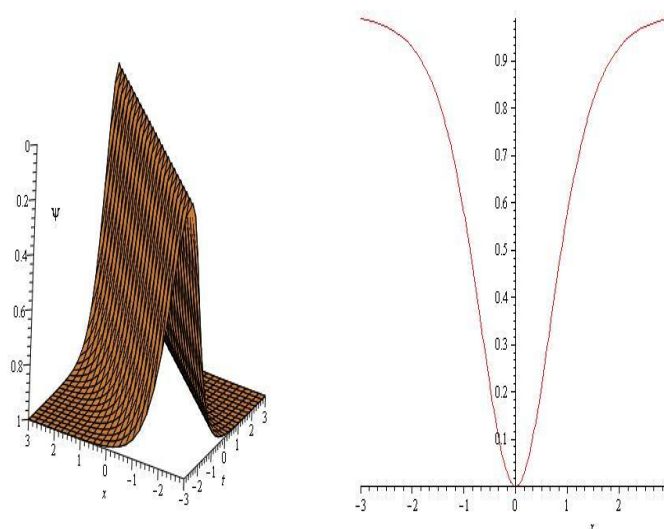

Fig. 2. Bell type soliton profile of Foam Drainage equation for $\mu = -1, k = 1, \ A = 0$ with $-3 \le x, t \le 3$. (Only shows the shape of $\psi_2(x,t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.
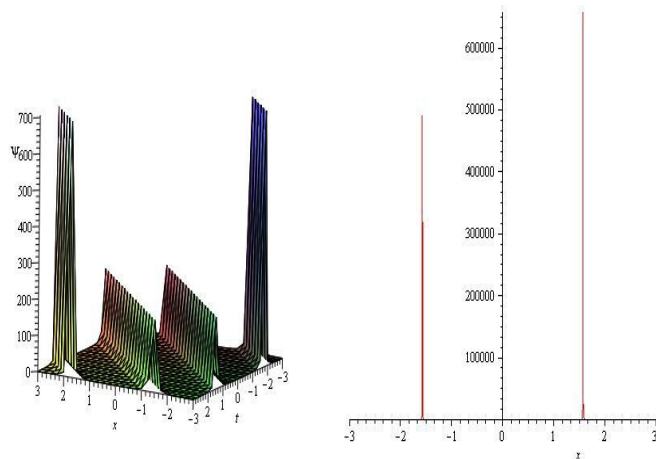
Fig. 3: Periodic type soliton profile of Foam Drianage equation for $\mu = 1, k = 1, \quad A = 0$ with $-3 \le x, t \le 3$. (Only shows the shape of $\psi_{10}(x,t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

Fig. 4: Singular soliton profile of Foam Drianage equation for $\mu = -1, k = 1, \quad A = 0$ with $-3 \le x, t \le 3$. (Only shows the shape of $\psi_5(x,t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.
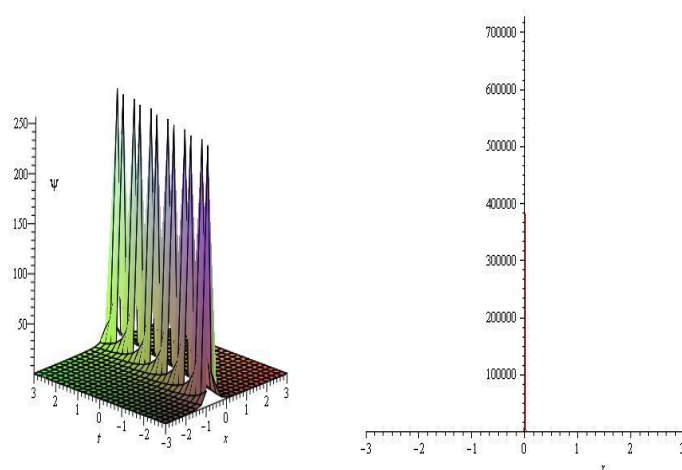
5. Conclusion:

Foaming occurs in many cleansing and amalgamation processes. The drainage of fluid foams involves the dealings of gravity, surface tension and viscous forces. In this paper, we use the enhanced (G'/G)-expansion method to seek the traveling wave solution of the foam drainage equation. The enhanced (G'/G)-expansion method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

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